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Statistical Mechanics of Nonlinear Wave Equations. 3. Metric Transitivity for Hyperbolic Sine-Gordon

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McKean and Vaninsky proved that the canonical measure $e^{-H}\,d^\infty Q\,d^\infty P$ based upon the Hamiltonian $H=\int[\frac{1}{2}P^2+\frac{1}{2}(Q')^2+F(Q)]\,dx$ of the wave equation $\partial^2Q/\partial t^2-\partial^2Q/\partial x^2+f(Q)=0$ with restoring force $f(Q)=F'(Q)$ is preserved by the associated flow of Q and $P=Q^\bullet$, and they conjectured that metric transitivity prevails, *always* on the whole line, and likewise on the circle *unless* $f(Q)=Q$ or $f(Q)=\text{sh}\,Q$. Here, the metric transitivity is proved for the whole line in the second case. The proof employs the beautiful "d'Alembert formula" of Krichever.

KEY WORDS: Partial differential equations; statistical mechanics; ergodic theory.

McKean and Vaninsky[5] discussed the petit ensemble for the nonlinear wave equation $\partial^2Q/\partial t^2-\partial^2Q/\partial x^2+f(Q)=0$, $f(Q)$ being an odd restoring force, i.e., it is of the same signature as Q. The data Q and $P=Q^\bullet$, taken at $t=0$, are distributed according to the Gibbsian canonical measure

$$e^{-H}\,d^\infty P\,d^\infty Q=e^{-(1/2)\int[P^2+(Q')^2]}\,d^\infty P\,d^\infty Q\times e^{-\int F(Q)}$$

in which $F(Q)$ is the primitive of $f(Q)$ and H is the Hamiltonian $\frac{1}{2}\int[P^2+(Q')^2]+\int F(Q)$ of the flow

$$Q^\bullet=P=\partial H/\partial P, \qquad P^\bullet=Q''-f(Q)=-\partial H/\partial Q$$

The meaning of the measure is easily explained. The factor $[\exp^{(-(1/2)\int P^2)}]\,d^\infty P$ states that P is white noise. As to $\{\exp^{[-(1/2)\int(Q')^2]}\}\,d^\infty Q$, think first of the circle $0\leqslant x<L$, i.e., let Q (and also P) be of period L. Then $\{\exp^{[-(1/2)\int(Q')^2]}\}\,d^\infty Q$ signifies that Q is

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"circular" Brownian motion, i.e., it is the standard Brownian motion starting at $Q(0)=h$, conditioned so as to be periodic $[Q(L)=h]$, the common level h being distributed over the line by the measure dh. The infinite total mass of this measure is tempered by the factor $e^{-\int F(Q)}$: in fact, if $\int_0^\infty e^{-F(h)}\,dh<\infty$ as for $f(Q)=\text{sh}\,Q$, then

$$Z=\int e^{-(1/2)\int(Q')^2}e^{-\int F(Q)}\,d^\infty Q<\infty$$

The distribution of Q may be made more transparent by a little trick: $F(\infty)=+\infty$, so $-(1/2)\,d^2/dQ^2+F(Q)$ has positive ground state ψ, with $\int\psi^2(\theta)\,dQ=1$ and eigenvalue Λ, in terms of which $F-\Lambda=(1/2)(m'+m^2)$ with $m=\psi'/\psi$. Now compute, by rules of the Brownian differential calculus, the (vanishing) integral of $d\lg\psi[Q(x)]$ over one period $0\leqslant x<L$: one has $d\lg\psi=m\,dQ+(1/2)\,m'(dQ)^2$ and $(dQ)^2=dx$, whence

$$0=\int m\,dQ+\tfrac{1}{2}\int m'\,dx=\int m\,dQ-\tfrac{1}{2}\int m^2\,dx+\int F\,dx-\Lambda L$$

and

$$e^{-(1/2)\int(Q')^2}e^{-\int F(Q)}=e^{-(1/2)\int(Q')^2}e^{\int m(Q)\,dQ-(1/2)\int m^2(Q)\,dx}$$

up to the unimportant factor $\exp(\Lambda L)$, which may be ignored. Here, one recognizes the law of the (circular) diffusion with infinitesimal operator $\mathscr{G}=(1/2)\,\partial^2/\partial Q^2+m(Q)\,\partial/\partial Q$ in which the odd function $m(Q)$ acts as a restoring drift, of signature opposite to that of Q, and it comes as no surprise that, as $L\uparrow\infty$, this law tends to that of the stationary diffusion with the same infinitesimal operator and stationary density $\psi^2(Q)$. It is in these ensembles that McKean and Vaninsky[5] established the existence of the flow and the invariance of the measure under it. They conjectured that the flow is metrically transitive: *always* in the case of the line, and likewise for the circle *unless* $f(Q)=m^2Q$ or $f(Q)=a\,\text{sh}(bQ)$, i.e., except for Klein/sinh-Gordon. The conjecture has a simple proof for sinh-Gordon on R. This is reported below, with further comments on Klein–Gordon. The rest is still open.

Step 1 notes that, for any wave equation, the data $Q_\pm=[Q(\pm x,x)$: $x\in R]$ on the characteristics $t=\pm x$ determine the whole solution, as is well known for classical solutions and carries over to the unpleasant data $H^0\times H^{-1}$ of the petit ensemble.

Step 2 is to observe that Q_+ and Q_- are copies of the horizontal diffusion $Q_0=[Q(0,x)$: $x\in\mathbb{R}]$ regulated by the infinitesimal operator $\mathfrak{G}$.

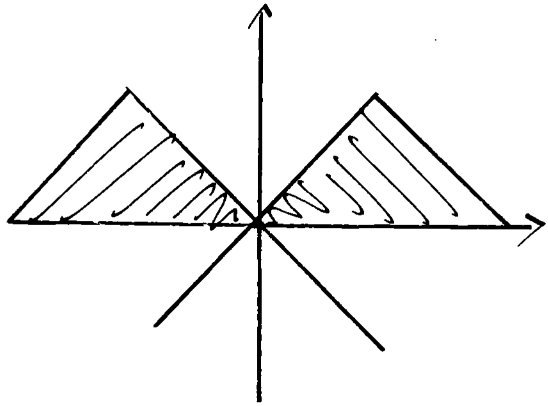

Fig. 1.

The same is true for any line $t = a + bx$ making an angle of $\leqslant 45°$ with the horizontal and has nothing to do with $f(Q) = \text{sh}\, Q$, as will appear from the proof.

Proof. The petit ensemble is invariant under space/time translations, so $[Q_+(x)\colon x \leqslant x_0]$, $[Q_+(x)\colon x \geqslant x_0]$, and $Q_+(x_0)$ stand in the same statistical relation as $[Q_+(x)\colon x \leqslant 0]$, $[Q_+(x)\colon x \geqslant 0]$, and $Q_+(0)$. But of these last three, the first/second is measurable over the field of $[P_0(x), Q_0(x)\colon x \leqslant 0]$, resp., $[P_0(x), Q_0(x)\colon x \geqslant 0]$, so they are independent, conditional upon $Q_+(0)$ (see Fig. 1), with the result that Q_+ itself is a (stationary) diffusion. Now $dQ_0 = dB + m(Q_0)\, dx$ with a free Brownian motion B starting at $B(0) = 0$, so, for $x \downarrow 0$,[2]

$$\begin{aligned} Q_+(x) &= Q_+(0) + \tfrac{1}{2} Q_0(2x) - \tfrac{1}{2} Q_0(0) + \tfrac{1}{2} \int_0^{2x} P_0(x')\, dx' + \tfrac{1}{2} \int_\Delta \text{sh}\, Q\, dt'\, dx' \\ &= Q_+(0) + \tfrac{1}{2} B(2x) + \tfrac{1}{2} \int_0^{2x} P_0(x')\, dx' + \tfrac{1}{2} \int_0^{2x} m(Q_0)\, dx' + O(x^2) \\ &= Q_+(0) + B_+(x) + m[Q_+(0)]\, x + o(x) \end{aligned}$$

in which the free Brownian motion $B_+(x) = (1/2)\, B(2x) + (1/2) \int_0^{2x} P_0$ is independent of the past $Q_+(x')\colon x' \leqslant 0$; compare Fig. 1. The rest will be plain.

Step 3 recalls the analog for sinh-Gordon of d'Alembert's formula for the free wave equation; it is due to Krichever.[(3)] We express

[2] Δ signifies the triangle with vertices 00, xx, $2x0$.

$\partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 + \operatorname{sh} Q = 0$ in light-cone coordinates $\xi = \frac{1}{2}(x+t)$ and $\eta = \frac{1}{2}(x-t)$. It takes the form $\partial^2 Q/\partial\xi\,\partial\eta = 4 \operatorname{sh} Q$, which is equivalent to the compatibility[3] of

$$\frac{\partial\psi}{\partial\xi}\psi^{-1} = \frac{1}{2}\frac{\partial Q}{\partial\xi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ \lambda^{-1} & 0\end{pmatrix} \quad \text{and} \quad \frac{\partial\psi}{\partial\eta}\psi^{-1} = \begin{pmatrix}0 & \lambda e^{Q}\\ e^{-Q} & 0\end{pmatrix}$$

for the function $\psi\colon (\xi, \eta, \lambda) \to SL(2, C)$ specified by the condition $\psi = 1$ at $\xi = \eta = 0$. Here ψ is an analytic function of λ in the twice-punctured sphere $\mathbb{P} - 0 - \infty$. Write $\psi = R_0^{-1} S_\infty$, R_0 being analytic in $\mathbb{P} - \infty$, with value $\left(\begin{smallmatrix}1 & 0\\ * & 1\end{smallmatrix}\right)$ at $\lambda = 0$, and S_∞ analytic in $\mathbb{P} - 0$, with value $\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)$ at $\lambda = \infty$. This factorization can be made in one and only one way; also, both pieces have determinant 1, necessarily. What is remarkable is that S_∞ is independent of η: indeed,[4]

$$\frac{\partial S_\infty}{\partial\eta} S_\infty^{-1} = \frac{\partial R_0}{\partial\eta} R_0^{-1} + R_0 \begin{pmatrix}0 & \lambda e^{Q}\\ e^{-Q} & 0\end{pmatrix} R_0^{-1}$$

is analytic on the whole sphere $\mathbb{P}$: as such, it is constant as regards $\lambda \in C$ and reduces to $\left(\begin{smallmatrix}0 & 0\\ * & 0\end{smallmatrix}\right)$ at 0 and to $\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)$ at ∞, so it must vanish identically. S_∞ is now determined, from Q_+ alone, by the rule

$$\frac{d}{dx} S_\infty(x, 0)\, S_\infty^{-1}(x, 0) = \frac{1}{2} Q'_+(x) \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ \lambda^{-1} & 0\end{pmatrix}$$

The game can be played the other way around: write $\psi = R_\infty^{-1} S_0$, S_0 being analytic in $\mathbb{P} - \infty$ and R_∞ analytic in $\mathbb{P} - 0$, with the same normalizations at 0 and ∞ as before. Now[5]

$$\frac{\partial S_0}{\partial\xi} S_0^{-1} = \frac{\partial R_\infty}{\partial\xi} R_\infty^{-1} + R_\infty \left[\frac{1}{2}\frac{\partial Q}{\partial\xi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ \lambda^{-1} & 0\end{pmatrix}\right] R_\infty^{-1}$$

vanishes for like reasons, and S_0 is determined, from Q_- alone, by the rule

$$\frac{d}{dx} S_0(0, x)\, S_0^{-1}(0, x) = \begin{pmatrix}0 & \lambda e\\ e^{-1} & 0\end{pmatrix} \quad \text{with} \quad e = \exp[Q_-(x)]$$

Also,

$$-\frac{\partial R_\infty}{\partial\xi} = R_\infty \left[\frac{1}{2} Q' \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right] \quad \text{at} \quad \lambda = \infty \quad \text{with} \quad Q' = \frac{\partial Q}{\partial\xi}$$

[3] This means $\partial^2\psi/\partial\xi\,\partial\eta = \partial^2\psi/\partial\eta\,\partial\xi$.

[4] Use $\partial\psi/\partial\eta\,\psi^{-1} = \left(\begin{smallmatrix}0 & \\ e^{-Q} & \text{etc.}\end{smallmatrix}\right)$.

[5] Use $\partial\psi/\partial\xi\,\psi^{-1} = \frac{1}{2}\partial Q/\partial\xi \left(\begin{smallmatrix}1 & \\ 0 & \text{etc.}\end{smallmatrix}\right)$.

so knowledge of R_∞ at ∞ permits one to recover the full solution $Q(t, x)$ from $Q_-(x)$ since $2\lg r_{11} + Q$ does not depend upon ξ. This is not all! $S_\infty S_0^{-1} = R_0 R_\infty^{-1}$ and the left side determines both factors on the right side separately,[6] and so also Q from Q_- and Q_+. This is "d'Alembert's formula," reducing the solution of $\partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 + \operatorname{sh} Q = 0$ to (a) determining S_∞/S_0 from Q_+/Q_-, (b) refactoring $S_\infty S_0^{-1}$ as $R_0 R_\infty^{-1}$, and (c) extracting Q from R_∞ and Q_-.

Warning. The determination of S_∞ from Q_+ assumes that Q'_+ exists, which is not true in the the petit ensemble. This is easy to fix: $dQ_+ = dB + m(Q_+)\,dx$ with a standard Brownian motion B, and $S_\infty = S_\infty(x, 0)$ is *the* nonanticipating solution of

$$S_\infty = 1 + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_0^x S_\infty\, dQ_+ + \begin{pmatrix} 0 & 1 \\ \lambda^{-1} & 0 \end{pmatrix} \int_0^x S_\infty\, dx'$$

$S_\infty\, dB$ being interpreted with dB *centered*, i.e., with

$$\begin{aligned}\int_0^x S_\infty\, dB &= \lim_{n\uparrow\infty} \sum_{k/n \leqslant x} S_\infty\left(\frac{k}{n}\right)\left[B\left(\frac{k+1/2}{n}\right) - B\left(\frac{k-1/2}{n}\right)\right] \\ &= \lim_{n\uparrow\infty} \sum_{k/n \leqslant x} S_\infty\left(\frac{k}{n}\right)\left[B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right] + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_0^x S_\infty\end{aligned}$$

Line 2 is the "nonanticipating" mode of writing with the differential in the future, so to say, and correction $\frac{1}{4}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \int_0^x S_\infty$ arising from the rule $(dB)^2 = dx$; see McKean[(4)] for such matters. S_∞ is determined in this way, with probability 1 in the petit ensemble, and the "d'Alembert solution" so produced solves the wave equation in its customary integral form:

$$Q(t, x) = \tfrac{1}{2}[Q_0(x - t) + Q_0(x + t)] + \tfrac{1}{2}\int_{x-t}^{x+t} P_0(x')\, dx' + \tfrac{1}{2}\int_\Delta \operatorname{sh} Q\, dt'\, dx'$$

that is the best one could expect.

Step 4. Now subject the random field $[Q(t, x)\colon (t, x) \in R^2]$ to the vertical shift $Q(t, x) \to Q(t + T, x)$. Then $S_\infty \to S_\infty(\bullet + T/2)\, S_\infty^{-1}(T/2) \equiv S_\infty^{T/2}$,[7] and $S_0 \to S_0(\bullet - T/2)\, S_0^{-1}(-T/2) \equiv S_0^{-T/2}$. But $S_\infty^{T/2}$, resp. $S_0^{-T/2}$, is determined by $Q_+(\bullet + T/2)$, resp. $Q_-(\bullet - T/2)$. The latter shifts are, individually, metrically transitive and even mixing—and more: $Q_+(x + T/2)$ and $Q_-(x - T/2)$ are independent, conditional on $Q(0)$, as soon as $T/2 \geqslant |x|$, as

[6] Use $R_0 = \left(\begin{smallmatrix} 1 & 0 \\ \bullet & 1 \end{smallmatrix}\right)$ at 0 and $R_\infty = \left(\begin{smallmatrix} \bullet & \bullet \\ 0 & \bullet \end{smallmatrix}\right)$ at ∞.

[7] The normalization $S_\infty(0) = 1$ must be respected.

can be seen from Fig. 1. Any residual dependence due to $Q(0)$ washes out for $T\uparrow\infty$, so that the joint shift, and also the flow $S_\infty S_0^{-1}\to S_\infty^{T/2}S_0^{-T/2}$, is mixing, too, and Q inherits this property via d'Alembert's formula. The proof is finished.

Klein–Gordon (with mass m) illustrates some finer points which have not been verified otherwise, even for sinh-Gordon. Now $\Box Q+m^2Q=0$, P is white, as before, and Q is the (Gaussian) Ornstein–Uhlenbeck process with mass m, infinitesimal operator $(1/2)\,\partial^2/\partial Q^2-mQ\,\partial/\partial Q$, and correlation $(2m)^{-1}\exp(-m\,|x|)$. The correlation of the field $Q(t,x)$ is easily found from

$$Q(t,x)=\cos(t\Delta)\,Q_0(x)+\sin(t\Delta)\,\Delta^{-1}P_0(x)\qquad\text{with}\quad \Delta=(m^2-D^2)^{1/2}$$

$\Delta^{-1}P_0$ is an independent copy of Q_0, so[8]

$$\begin{aligned}E[Q(t,x)\,Q(0)]&=[\Delta^{-2}\cos t\Delta](x,0)\\&=\frac{1}{2\pi}\int\frac{\cos t(k^2+m^2)^{1/2}}{k^2+m^2}\,e^{(-1)^{1/2}kx}\,dk\\&=\frac{e^{-m\,|x|}}{2m}-\frac{1}{2m}\int_{|x|}^{|t|}J_0(m[(t')^2-x^2]^{1/2})\,dt'\end{aligned}$$

with the understanding that the integral is present only if $|x|<|t|$; in particular, it is absent if $t=\pm cx$ and $|c|\leqslant 1$, confirming the result of step 2. The process $Q_\uparrow=Q(\bullet,0)$ is of special interest[9]:

$$E[Q_\uparrow\otimes Q_\uparrow]=\frac{1}{2m}-\frac{1}{2m}\int_0^t J_0(mt')\,dt'=\frac{1}{\pi}\int_m^\infty\frac{\cos tk}{(k^2-m^2)^{1/2}}\frac{dk}{k}$$

from which follows the curious fact that the past $Q_\uparrow(t)$: $t\leqslant 0$ determines the future $Q_\uparrow(t)$: $t\geqslant 0$ since the spectral weight omits a band; also, mixing follows from the vanishing of $E[Q_\uparrow\otimes Q_\uparrow]$ for $t\uparrow\infty$.[10] $P_\uparrow=Q'(\bullet,0)$ is an independent copy of $(-D^2-m^2)\,Q_\uparrow$ and shares its determinism/mixing in view of

$$E[P_\uparrow\otimes P_\uparrow]=\frac{1}{\pi}\int_m^\infty\cos tk(k^2-m^2)^{1/2}\frac{dk}{k}$$

[8] J_0 is the standard Bessel function; see Bateman [ref. 1, 26(30)] for the necessary transform.

[9] $Q\otimes Q$ means $Q(t_1)\,Q(t_2)$; also $t=|t_2-t_1|$.

[10] See, e.g., Dym and McKean[(2)] for such matters.

It is noteworthy that the "vertical" ensemble for $P_\uparrow$ and $Q_\uparrow$ so produced is invariant under the horizontal flow despite the fact that $f(Q) = m^2 Q$ now acts as a *repulsive* force: one does not expect a finite invariant measure then. The mystery is resolved by noting that the vertical ensemble is not of Gibbs type, i.e., unlike the "horizontal" ensemble, it has no mechanical interpretation.

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